STEM Scholars Program Topic: The Search for Boxes with Integer Dimensions

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This project will look at geometric figures with (positive) integer dimensions. In particular, we will try to construct an object which may or may not exist.

Suppose you wanted to construct a rectangle in a way where each side has integer length, and each diagonal has integer length. Can this be done? The answer is "yes", and there are many solutions.

The simplest example is to take a 3×4 rectangle, producing a diagonal of length 5. But there are many other rectangles with this property, for example a 180×2021 rectangle has diagonal length 2029.

Once such rectangles are known to exist, it is natural to ask if we have an effective way to describe them all. We do: the dimensions can be described in terms of three positive integers r, s, t with s, t subject to mild, easy-to-verify restrictions. (We will see how this works early on in the project.)

While presented as a geometry problem, it can also be viewed as a problem from number theory. Number theory is the study of integer solutions to equations. Here, if two adjacent sides of the rectangle have length x and y then the diagonal is $\sqrt{x^2 + y^2}$. Thus the problem is to choose x and y so that $\sqrt{x^2 + y^2}$ is an integer; equivalently we wish to find integer solutions to the equation

$$x^2 + y^2 = z^2.$$

Now suppose we want to be more ambitious. We ask: is there a (three dimensional) rectangular box with integer dimensions? Here the dimensions we are interested in are length, width, height, diagonals along each face, and diagonals through the solid itself.

This is an unsolved problem, and can be viewed as finding integer solutions to a system of four equations, three unknowns.

Here are some of the things we can do.

1. We shall try to prove whether or not such a rectangular box exists.

2. If we prove such boxes exist, we shall try to find a way to describe all such boxes.

3. If we prove that such boxes cannot exist, we will be mathematically famous.

4. If we cannot prove whether such boxes exist, we will find conditions on the dimensions for such a box to exist. For example, perhaps we could show that each side has to have length at least 100, or that each side has to be of odd length, or that the box cannot be a cube.

5. Any serious investigation of the integer box problem requires a study of the integer rectangle problem. We could ask questions here as well. For example, I provided an example above where one of the sides has length 2021. Are there others? If so, how many? More generally, for a given x, how many integer rectangles can be found with one side of length x? Does every positive integer appear as a side length?

6. Time permitting, we could try to construct other integer dimension objects. These would probably be plane figures instead of solids, for example other quadrilaterals or polygons with an even number of sides. To give one natural generalization of the (solved) rectangle problem, a general parallelogram has two diagonals of different length: we could try to determine when *both* diagonals are of integer length.

Mathematical research is unlike what you see in the classroom. It should be thought of as a puzzle that we will work through together. And, since it is *our* puzzle, we can change the parameters as we see fit; what I offer above is just a few of the directions we could go, but ultimately our journey is entirely up to us.